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LAMINAR FLOW AND HEAT TRANSFER NEAR ROTATING AXISYMMETRIC SURFACE

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Аннотация—Приводятся результаты расчетов на электронной вычислительной машине подобных пограничных слоев на осесимметричных поверхностях, вращающихся в неограниченной неподвижной среде. Условием существования подобных слоев является степенная зависимость от длины образующей расстояния от оси вращения.

Рассчитаны характеристики скоростного и теплового пограничных слоев, распределения скоростей и температур.

При помощи полученного класса точных решений строится приближенный метод расчета скоростного и теплового пограничных слоев на вращающихся поверхностях произвольной формы путем подбора на каждом участке близкой поверхности со степенной зависимостью радиуса от длины образующей с учетом непрерывного сращивания пограничного слоя.

На примере вращающейся сферы показано, что результаты расчетов этим методом согласуются с данными других расчетов и с экспериментами.

NOMENCLATURE

x,	distance along the generating line;
y,	transverse direction;
z,	distance along the normal from a surface;
r,	distance from rotation axis;
r_m,	maximum radius of a surface;
x^0,	distance along rotating axis;
r_0,	initial (or finite) surface radius [by formula (9)]; $x^0_* = x^0 : A^{1/1-m}$, $r_* = r : A^{1/1-m}$, $\bar{x} = x : r_m$
$x_1\,(x_2)$; $r_1\,(r_2)$,	co-ordinates of the beginning (end) of surface section;
x_0,	constant;
A,	constant;
m,	exponent;
$\beta =$	$(1 + 3m)/4m$;
u, v, w,	components of velocity vector in x, y, z-direction;
T,	temperature;
T_w,	surface temperature;
T_∞,	medium temperature;
ω,	angular velocity;
ψ,	stream function;
$\dot{r} =$	$\mathrm{d}r/\mathrm{d}x$;
$H(\zeta)$; ζ; $F(\zeta)$; $G(\zeta)$,	defined by formulae (3);
$\tau(\zeta)$; $\sigma(\zeta)$,	dimensionless temperature defined by formula (12);
$\sigma_i(\zeta)$,	dimensionless temperature at thermally insulated surface;
	$\sigma_y^* = \int\limits_0^\infty \frac{v}{r\omega}\,\mathrm{d}z$;
	$\Delta_y = \int\limits_0^\infty \left(\frac{v}{r\omega}\right)^2 \mathrm{d}z$;
	$\delta_t = \int\limits_0^\infty \tau_t\,\mathrm{d}z$;
C_p,	heat capacity at constant pressure;
ρ,	density;
ν,	kinematic viscosity;
λ,	heat conduction;

τ_x, τ_y, components of shear stress on a surface;
q, specific heat flow;
$\bar{q}$, mean specific heat flow;
τ_x^0; τ_y^0; q^0; $\bar{q}^0$, disk values at the same quantities of r, ω and physical constants;
A, Z, defined by formula (20);

$$C(\beta) = \int_0^\infty G^2 \, d\zeta;$$

$$\theta = \frac{r^2 \omega^2}{C_p (T_w - T_\infty)};$$

C_m, coefficient of moment of shear stress;

$$C_m = \frac{4\pi}{\rho} \frac{\int_0^{x_m} r^2 \tau_y \, dx}{r_m^4 \, \nu^{1/2} \, \omega^{3/2}};$$

$Pr = \frac{\rho\nu C_p}{\lambda}$, Prandtl number;

$Nu = \frac{qr}{\lambda(T_w - T_\infty)}$, Nusselt numbers;

$$\overline{Nu} = \frac{\bar{q}r}{\lambda(T_w - T_\infty)};$$

Nu^0, Nusselt number for a disk;

$$Re = \frac{r^2 \omega}{\nu};$$

$Re_m = \frac{r_m^2 \omega}{\nu}$, Reynolds number.

INTRODUCTION

PROBLEMS of calculating the laminar flow and heat transfer near axisymmetric rotating surfaces arise in turbine construction and in other technical fields. Detailed investigations are conducted for the case of a rotating disk and cylinder (see [1]). Only integral methods are developed for surfaces of other shapes. For a sphere these are worked out by Howard [2] and Nigam [3], and for an arbitrary surface, by the author [4]. Baxter and Davies [5] have calculated heat transfer close to a rotating spherical surface. The author [4] has proposed an approximate method for calculating heat transfer for the arbitrary surfaces. Calculations turn out to be rather complex as they should satisfy a greater number of integral relations than in the case of the plane flow. For this purpose it is therefore reasonable to use similar exact solutions of boundary-layer equations which are, in themselves of particular interest too.

SOLUTION OF A VELOCITY BOUNDARY-LAYER EQUATION

Consider the equations of a laminar boundary layer forming on an axisymmetric surface which rotates with a constant angular velocity in an infinite motionless medium

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} - \frac{v^2}{r}\frac{dr}{dx} + w\frac{\partial u}{\partial z} &= \nu\frac{\partial^2 u}{\partial z^2} \\ u\frac{\partial v}{\partial x} + \frac{uv}{r}\frac{dr}{dx} + w\frac{\partial v}{\partial z} &= \nu\frac{\partial^2 v}{\partial z^2} \\ \frac{1}{r}\frac{\partial(ur)}{\partial x} + \frac{\partial w}{\partial z} &= 0 \end{aligned}\right\} \quad (1)$$

Geis [6] has shown that the similar solutions of system (1) exist only in the case when $r(x)$ is a power function of $(x + x_0)$

$$r = A\,(x + x_0)^m \quad (2)$$

In actual fact, if the stream function ψ is introduced so that $(\partial\psi/\partial z) = ru$; $(\partial\psi/\partial x) = -rw$ and if we take for $m > 0$ (i.e. $\dot{r} > 0$)

$$\left.\begin{aligned} &\zeta = z\sqrt{\dot{r}\left(\frac{\omega}{\nu}\right)},\ \psi = -\frac{r^2}{2}\sqrt{\left(\frac{\nu\omega}{\dot{r}}\right)}\,H(\zeta) \\ &u = r\omega\,F(\zeta), \qquad v = r\omega\,G(\zeta) \end{aligned}\right\} \quad (3)$$

then, equation (1) is reduced to the system of the ordinary differential equations

$$\left.\begin{aligned} F'' &= F^2 - G^2 + \beta HF' \\ G'' &= 2FG + \beta HG' \\ H' + 2F &= 0 \end{aligned}\right\} \quad (4)$$

where

$$\beta = \frac{1 + 3m}{4m} \quad (5)$$

The boundary conditions will be

$$F(0) = H(0) = 0, \quad G(0) = 1, \quad F(\infty) = G(\infty) = 0 \tag{6}$$

The expression for w becomes

$$\frac{w}{\sqrt{(\nu\omega\dot{r})}} = \beta H + 2(\beta - 1) F(\zeta) \tag{7}$$

The shape of the generating line of the surface which corresponds to (2) is defined by the equations

$$\left. \begin{aligned} x^0 &= \int_0^x \sqrt{[1 - m^2 A^2 (x + x_0)^{2(m-1)}]}\, \mathrm{d}x \\ r &= A\,(x + x_0)^m \end{aligned} \right\} \tag{8}$$

At $0 < m \leqslant 1$ the integrand takes real values, starting with $x_0 = (mA)^{1/1-m}$, that corresponds to the initial radius

$$r_0 = m^{m/1-m} A^{1/1-m} \tag{9}$$

and then the surface radius increases with x. At $m > 1$ ($\frac{3}{4} < \beta < 1$) the surface starts with a zero radius and finishes at the radius defined by formula (9). From formula (7) it follows that all the surfaces with the same m are similar between themselves

$$x_*^0 = f(r_*), \quad x_*^0 = x^0 : A^{1/1-m}, \quad r_* = r : A^{1/1-m} \tag{10}$$

Thus, the dimensionless motion equations (4) do not depend on A.

Note that close to the edges corresponding to the intial (or finite) radius r_0 of a surface, equations (1) are not valid. This also refers to very thin bodies when β approaches 3/4 or when A is small.

Boundary value problem (4), (6) was solved by the trial and error method together with interpolation. Since the problem is non-linear, it is necessary to prescribe rather exact initial values of $F'(0)$ and $G'(0)$. For $\beta = 1$ (case of a rotating disc or cone) they may be taken from the solution by Cochran [7]; for β close to 1 from the approximate solution of the problem obtained by the method of integral relations (Appendix 1), for other β values, by means of extrapolation using the values of $F'(0)$ and $G'(0)$ found earlier.

The difficulty due to the integration range extending to infinity is overcome taking into account the fact that the unknown functions, starting with finite value $\zeta = \zeta^*$ ($\zeta^* \geqslant 12$) in fact do get values at infinity, i.e.

$$F(\zeta^*) = G(\zeta^*) = 0.$$

Having accounted for $\zeta^* = 12$, we find the solution for the increased values of ζ^* and if the result does not change, then we finish the process of successive approximation.

System (1) is integrated by the Merson modification of the Runge–Kutta method [8] with accuracy $\epsilon = 10^{-7}$ at each step. The conditions $F(\zeta^*) = G(\zeta^*) = 0$ are satisfied with the same degree of accuracy.

Integral characteristics $\int_0^\infty G\,\mathrm{d}\zeta$, $\int_0^\infty G^2\,\mathrm{d}\zeta$ of a

Table 1.

β	$F'(0)$	$-G'(0)$	$-H(\infty)$	$\int_0^\infty G\,\mathrm{d}\zeta$	$\int_0^\infty G^2\,\mathrm{d}\zeta$
1	0·510233	0·615922	0·884473	1·27144	0·672527
2	0·465073	0·654174	0·579152	1·11620	0·609756
3	0·434162	0·688635	0·443089	1·02530	0·567934
4	0·411243	0·719243	0·363984	0·962485	0·537274
5	0·393277	0·746662	0·311545	0·915179	0·513388
6	0·378632	0·771498	0·273912	0·877614	0·493996
7	0·366352	0·794224	0·245425	0·846681	0·477778
8	0·355829	0·815201	0·223014	0·820534	0·463910
9	0·346659	0·834704	0·204862	0·797985	0·451843
10	0·338558	0·852950	0·189819	0·778229	0·441196

boundary layer are defined whilst calculating these functions. To check the calculation the value $\int_0^\infty FG\,d\zeta$ is defined, and the relation is tested

$$\int_0^\infty FG\,d\zeta = -G'(0):2(1+\beta).$$

This relation is satisfied with accuracy up to 10^{-7}. The basic quantities which characterize a boundary layer are given in Table 1. Velocity charts are given in Fig. 1.

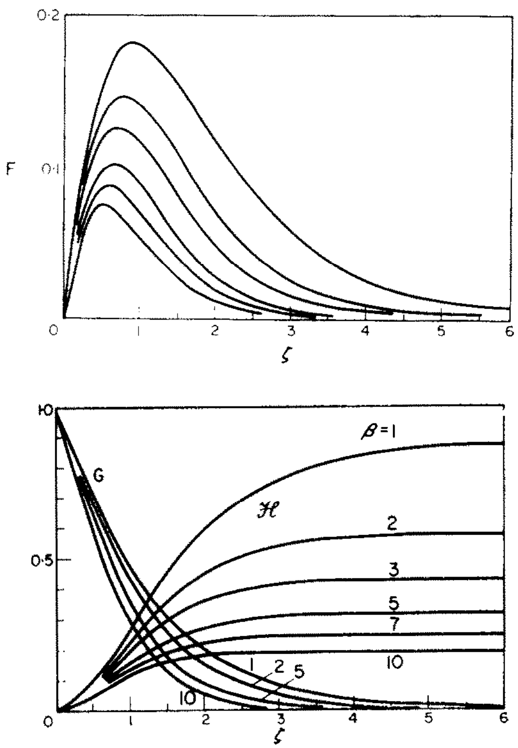

FIG. 1. Distribution of velocity vector components in a boundary layer.

SOLUTION OF A THERMAL BOUNDARY-LAYER EQUATION

The solution of the equation for a thermal boundary layer

$$u\frac{\partial T}{\partial x}+w\frac{\partial T}{\partial z}=\frac{\nu}{Pr}\frac{\partial^2 T}{\partial z^2}+\frac{\nu}{C_p}\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right] \quad (11)$$

is sought in the form

$$T = T_\infty + \frac{\mu\omega\nu}{\lambda}\tau(\zeta)+\frac{r^2\,\omega^2}{C_p}\sigma(\zeta) \quad (12)$$

Then, equation (11) is reduced to two ordinary differential equations

$$\frac{1}{Pr}\tau'' = \beta H\tau'$$

$$\frac{1}{Pr}\sigma'' = \beta H\sigma' + 2F\sigma' + 2F\sigma - (F'^2+G'^2) \quad (13)$$

For the isothermal surface the boundary conditions will be

$$\tau(0)=1,\quad \tau(\infty)=0,\quad \sigma(0)=\sigma(\infty)=0 \quad (14)$$

And

$$\frac{T-T_\infty}{T_w-T_\infty}=\tau(\zeta)+\frac{r^2\,\omega^2}{C_p\,(T_w-T_\infty)}\sigma(\zeta) \quad (15)$$

The second summand characterizes energy dissipation, whose intensity depends on the multiplier

$$\theta = \frac{r^2\,\omega^2}{C_p\,(T_w-T_\infty)} \quad (16)$$

For the surface thermally insulated the boundary conditions become

$$\tau'(0)=\tau(\infty)=0 \quad \sigma'(0)=\sigma(\infty)=0 \quad (17)$$

whence it follows that $\tau \equiv 0$.

Since equations (13) are linear, it is quite enough to have two samples with linear interpolation to solve these equations numerically.

Whilst calculating the unknown functions, the values of the integral characteristics of a boundary layer are determined:

$$\int_0^\infty \tau(\zeta)\,d\zeta$$

The value $\int_0^\infty F\tau\,d\zeta$ is also calculated to check the relation

$$\int_0^\infty F\tau \, d\zeta = -\tau'(0) : 2\beta \, Pr \qquad (18)$$

As a control the validity of identities (II-4) and (II-5) are checked (see Appendix II).

For illustration of high accuracy of the solutions of equations (13) and (4), some calculation results for $\beta = 1$ are compared with data of other calculations (Table 2).

The basic quantities characterizing a heat boundary layer are given in Tables 3 to 6. Figure 2 illustrates typical temperature distributions in a boundary layer.

Calculations show (Fig. 3) that a shape of a surface (parameter β) slightly influences the dependence of the heat-transfer coefficient upon the Prandtl number. The multiplier $Pr^{0.4}$ may reflect this dependence only in some Prandtl number range.

Table 2. Comparision with other calculations for $\beta = 1$ and $Pr = 0.7$

Data	Our calculation	Rogers, Lance [9]	Sparrow, Gregg [10]
$F'(0)$	0·510233	0·510233	0·510
$-G'(0)$	0·615922	0·615922	0·6159
$-H(\infty)$	0·88447	0·88446	0·8845
$\int_0^\infty G \, d\zeta$	1·27144	—	1·271
$\int_0^\infty G^2 \, d\zeta$	0·672527	—	0·6721
$-\tau'(0)$	0·323123	—	0·3231
$\int_0^\infty \tau \, d\zeta$	2·10581	—	2·106

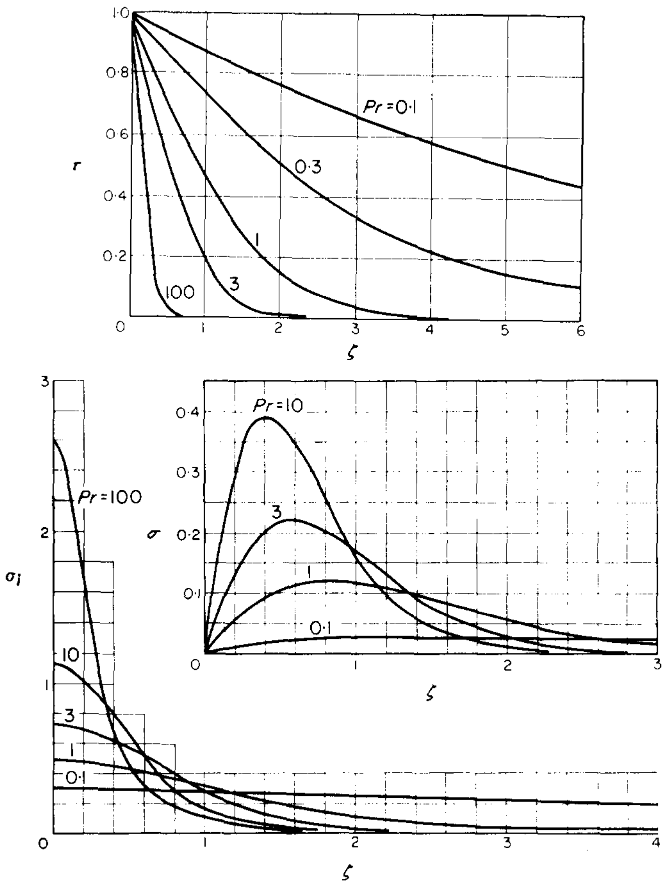

FIG. 2. Temperature distribution in a boundary layer at $\beta = 3$.

Table 3. Values of $-\tau'(0)$

β \ Pr	0·1	0·3	0·72	1	3	10	30	100
1	0·085154	0·185106	0·328573	0·396248	0·682580	1·13412	1·73103	2·68714
2	0·10489	0·237935	0·418609	0·502801	0·855853	1·40883	2·13836	3·30632
3	0·11759	0·271459	0·475551	0·570363	0·966671	1·58589	2·40221	3·70890
4	0·12744	0·296343	0·518130	0·620963	1·05008	1·71973	2·60219	4·01464
5	0·13574	0·316420	0·552567	0·661928	1·11783	1·82873	2·76535	4·26438
6	0·14261	0·333381	0·581714	0·696627	1·17535	1·92144	2·90428	4·47722
7	0·14836	0·348148	0·607126	0·726896	1·22561	2·00257	3·02594	4·66373
8	0·15375	0·361279	0·629745	0·753850	1·27042	2·07498	3·13461	4·83040
9	0·15895	0·373138	0·650192	0·778224	1·31099	2·14058	3·23311	4·98152
10	0·16324	0·383977	0·668895	0·800524	1·34813	2·20069	3·32341	5·12010

Table 4. Values of $\sigma'(0)$

β \ Pr	0·1	0·3	0·72	1	3	10	30	100
1	0·040334	0·109805	0·233695	0·307961	0·755363	1·93588	4·42031	10·6273
2	0·043097	0·117174	0·248721	0·327087	0·795482	2·01780	4·56399	10·8633
3	0·045489	0·123693	0·262063	0·344318	0·833965	2·10398	4·73332	11·1978
4	0·047577	0·129345	0·273852	0·359621	0·868949	2·18502	4·89890	11·5442
5	0·049427	0·134376	0·284386	0·373331	0·900661	2·25975	5·06084	11·8775
6	0·051102	0·138918	0·293913	0·385749	0·929590	2·32864	5·19984	12·1940
7	0·052639	0·143083	0·302622	0·397112	0·956187	2·39241	5·33536	12·4929
8	0·054047	0·146884	0·310654	0·407600	0·980818	2·45176	5·46220	12·7734
9	0·055345	0·15043	0·318119	0·417352	1·00378	2·50729	5·58133	13·0395
10	0·056572	0·153748	0·325100	0·426475	1·02530	2·55948	5·69370	13·2905

Table 5. Values of $\int_0^\infty \tau(\zeta)\,d\zeta$

β \ Pr	0·1	0·3	0·72	1	3	10	30	100
1	8·5713	4·0988	2·0641	1·6533	0·89266	0·51899	0·33486	0·21373
2	7·3960	3·1826	1·6032	1·2903	0·70723	0·41613	0·27040	0·17344
3	6·8092	2·7714	1·4059	1·1334	0·62467	0·36915	0·24048	0·15452
4	6·3885	2·5341	1·2879	1·0392	0·57435	0·34016	0·22189	0·14272
5	6·0421	2·3706	1·2062	0·97381	0·53914	0·31974	0·20874	0·13432
6	5·7963	2·2482	1·1448	0·92461	0·51250	0·30422	0·19871	0·12793
7	5·6307	2·1516	1·0963	0·88564	0·49130	0·29182	0·19070	0·12280
8	5·4580	2·0725	1·0564	0·85363	0·47384	0·28160	0·18407	0·11856
9	5·2693	2·0060	1·0229	0·82663	0·45909	0·27294	0·17845	0·11496
10	5·1629	1·9488	0·99399	0·80340	0·44636	0·26545	0·17359	0·11183

Table 6. Values of $\sigma_i(0)$

β \ Pr	0·1	0·3	0·72	1	3	10	30	100
1	0·27198	0·34812	0·44746	0·50000	0·75081	1·1946	1·8151	2·8381
2	0·29166	0·35835	0·45072	0·50000	0·73707	1·1563	1·7385	2·6910
3	0·30152	0·36331	0·45220	0·50000	0·73060	1·1378	1·7004	2·6159
4	0·30700	0·36591	0·45304	0·50000	0·72685	1·1270	1·6779	2·5707
5	0·31011	0·36757	0·45358	0·50000	0·72442	1·1199	1·6633	2·5406
6	0·31263	0·36872	0·45395	0·50000	0·72271	1·1149	1·6526	2·5192
7	0·31503	0·36957	0·45423	0·50000	0·72144	1·1112	1·6447	2·5032
8	0·31644	0·37022	0·45445	0·50000	0·72047	1·1084	1·6387	2·4907
9	0·31695	0·37073	0·45461	0·50000	0·71969	1·1061	1·6338	2·4808
10	0·31821	0·37115	0·45475	0·50000	0·71906	1·1042	1·6299	2·4727

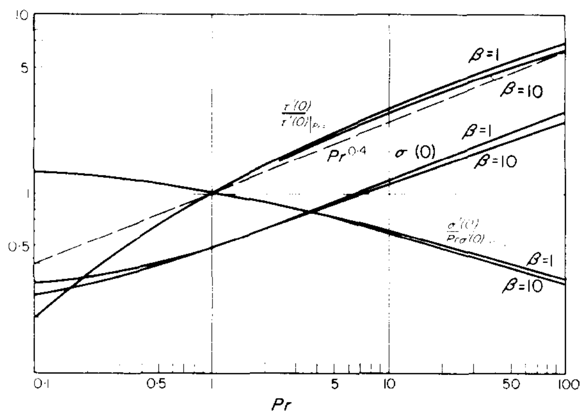

FIG. 3. Influence of the Prandtl number on heat transfer.

APPROXIMATE METHOD FOR SOLVING AN ARBITRARY ROTATING SURFACE

In order to approximately calculate a boundary layer on the arbitrary rotating surface with a monotonic increase in radius ($\dot r > 0$), we divide this surface into some small sections, choose a closely similar surface from class (2) for each surface. Calculation is based on change in the thickness of a boundary layer

$$\Delta_y = \int_0^\infty \left(\frac{v}{r\omega}\right)^2 \mathrm{d}z = C(\beta) \sqrt{\left(\frac{\nu}{\omega \dot r}\right)}$$

$$C(\beta) = \int_0^\infty G^2 \,\mathrm{d}\zeta \qquad (19)$$

If x_1 and x_2 are the ends of one of these sections,* then for the surface belonging to class (2) which passes through these points we have:

$$\frac{r_2}{r_1} = \left(1 + \frac{x_2 - x_1}{x_1 + x_0}\right)^m$$

Since $\dot r = mr : (x + x_0)$, then according to (19) $(x + x_0)^{-1} = C^2 \nu : \omega\, mr,\ \Delta^2_{y_1}$, so that designating

$$Z = \frac{x_2 - x_1}{\omega\, r_1 \Delta^2_{y_1}}, \quad A = \frac{r_2 - r_1}{x_2 - x_1}\left(\Delta^2_{y_1} \frac{\omega}{\nu}\right) \qquad (20)$$

we obtained the formula

$$A = [(1 + ZC^2 : m)^m - 1] : Z \qquad (21)$$

A set of curves $A(\beta)$ for different values of Z (Fig. 4) are constructed according to the known values of $C(\beta)$. If x_1, x_2 r_1 r_2 and value $[\Delta^2_y (\omega/\nu)]$ at a point x_1 are known, then Z and A may be calculated, and according to the curves $A(\beta; Z)$ the corresponding value of β and for it, C^2/m may be found. Thus, the value $[\Delta^2_y (\omega/\nu)]_2$ at a point x_2 is defined by the formula

$$\left(\Delta^2_y \frac{\omega}{\nu}\right)_2 = \left(\Delta^2_y \frac{\omega}{\nu}\right)_1 \frac{r_1}{r_2}(1 + ZC^2 : m) \quad (22)$$

All the remaining parameters of a boundary layer are determined by the parameter β. In particular, if the the values of Pr and θ are prescribed, heat transfer near a rotating surface

* For plane flows Smith [11] has proposed a similar method.

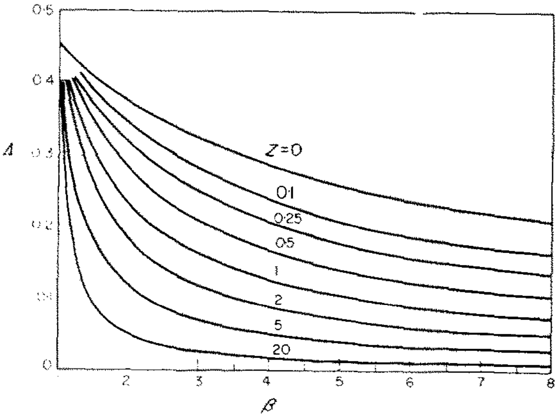

FIG. 4. Plots of β and Z vs Λ.

may be calculated. Besides Δ_y, we may estimate shear stress along the surface

$$\tau_x : \rho(r\omega)^2 = F'(0) \sqrt{\left(\frac{\nu \dot{r}}{r^2 \omega}\right)};$$

$$\tau_y : \rho(r\omega)^2 = G'(0) \sqrt{\left(\frac{\nu \dot{r}}{r^2 \omega}\right)} \tag{23}$$

the local heat-transfer coefficient

$$Nu = \frac{qr}{\lambda (T_w - T_\infty)} = - [\tau'(0) + \theta\, \sigma'(0)] \sqrt{\left(\frac{r^2 \omega}{\nu}\right)}\, \dot{r} \tag{24}$$

the displacement thickness δ_y^* as well as the thermal boundary-layer thickness δ_t

$$\sqrt{\left(\frac{\omega r}{\nu}\right)} \delta_y^* = \int_0^\infty G \, \mathrm{d}\zeta, \quad \delta_t \sqrt{\left(\frac{\omega \dot{r}}{\nu}\right)} = \int_0^\infty \tau \, \mathrm{d}\zeta \tag{25}$$

and distributions of temperatures and velocities in a boundary layer.

Thus, passing from one point to another it is possible to calculate the whole pattern of the development of a boundary layer on the arbitrary rotating surface at $\dot{r} > 0$.

CALCULATION EXAMPLE: ROTATING SPHERE

Calculation results for a rotating sphere are depicted in Fig. 5. Ratios of local values of components of shear stress along the surface for a radius $r(x)$ to the corresponding values for a disc with both the same radius and values ω, ρ, ν agree well with the calculation results obtained by the integral method [4].

Integration of a transverse component of shear stress gives the value of the coefficient $C_m = 3{\cdot}36$ [12]. It should be noted that as in Howarth's calculation [2] the separation of a boundary layer far from the equator supposed by Nigam's solutions [3] is not observed. This is confirmed by Bowden and Lord's [12] experiments. This calculation as well as others [2], [4] and experiments [13], [16] show that the boundary layer increases from poles to the equator.

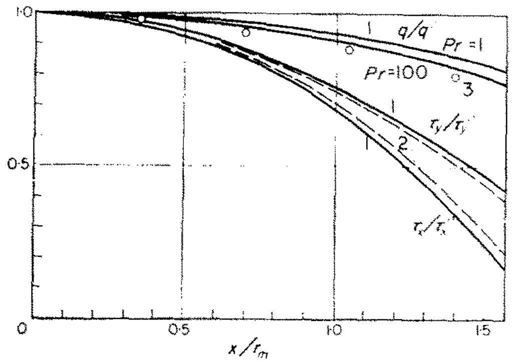

FIG. 5. Distribution of components of shear stress and heat flux along generating line of rotating sphere: 1. calculation by the present method; 2. by the method of integral relations [4]. 3. calculated by Baxter and Davies [5] for $Pr \to \infty$.

The calculation results (at $\theta = 0$) of the ratios between mean specific heat fluxes $\bar{q}/\bar{q}^0$ at $T_w - T_\infty = \text{const.}$ from $r = 0$ and the given sphere radius $r(x)$ to the appropriate disk values agree well with those obtained by Baxter and Davies [5]. The quantity $\bar{q}$ is defined by the formula:

$$\bar{q} = \frac{2\pi \int_0^{\bar{x}} q(x)\, r(x)\, \mathrm{d}x}{2\pi \int_0^{\bar{x}} r(x)\, \mathrm{d}x} = \frac{\int_0^{\bar{x}} q(\bar{x})\, r(\bar{x})\, \mathrm{d}\bar{x}}{1 - \cos \bar{x}}.$$

As it should be expected (Fig. 3) the ratios of the mean specific heat fluxes for the whole sphere to the appropriate values for a disk $\bar{q}_n : \bar{q}_m^0 = \overline{Nu} : \overline{Nu}^0$ slightly depend upon the Prandtl number (Table 7).

Table 7. Values of $\overline{Nu}/\overline{Nu}^0$ for a sphere

Pr	0·72	1	10	100	∞
$\dfrac{\overline{Nu}}{\overline{Nu}^0} = \dfrac{\bar{q}_m}{\bar{q}_m{}^0}$	0·843	0·840	0·822	0·815	0·75

Let us compare (Fig. 6) the results obtained with the experiments [13] for air ($Pr = 0{\cdot}72$). According to Tables 3 to 7 $\overline{Nu} = 0{\cdot}276\ Re^{0{\cdot}5}$, that is below the experimental data.

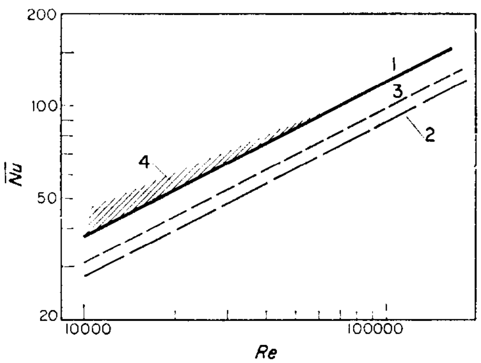

FIG. 6. Dependence of Nu upon Re_m for a sphere: 1. experiments of Kreith and his co-workers [13]; 2. theoretical dependence; 3. calculation using Cobb and Saunders' data [15]; 4. calculations using Young's data [14].

This discrepancy may be in part due to the effect of natural convection, especially with rotation about the vertical axis (despite the fact that experiments at small Grashof numbers were used). Therefore, if Young's experimental data [14] are used for $\overline{Nu}^0$ when the disk also rotates about the vertical axis, then a better agreement with the data by Kreith and his co-workers is achieved. The intermediate value is obtained by using the experimental formula of Cobb and Saunders [15] for $\overline{Nu}^0$ with the horizontal rotation axis.

Some intensification of heat transfer which is not taken into account in calculations may be expected from ejection of a jet at the equator. It is unknown, how much the isothermal condition of a sphere is maintained, since non-isothermal conditions may also influence heat transfer.

APPENDIX I

Approximate Calculation of Velocity Boundary Layer

We solve system (4)–(6) by the method of integral relations. We give the unknown functions in the form of polynomials in $t = \zeta/\zeta_0$, (where ζ_0 is the dimensionless thickness of a boundary layer), which satisfy the boundary conditions of the problem

$$F(0) = G(0) = 1,$$
$$F(1) = F'(1) = G(1) = G'(1) = 0,$$
$$G''(0) = 0, \qquad F''(0) = -1.$$

Then,

$$\left.\begin{aligned} F &= (1-t)^2(\lambda t + 2\lambda t^2 - \tfrac{1}{2}t^2)\zeta_0^2 \\ G &= \tfrac{1}{2}(2+t)(1-t)^2 \end{aligned}\right\} \tag{I-1}$$

Upon integrating equation (4) along a boundary layer, the integral relations are obtainable:

$$\left.\begin{aligned} -F'(0) &= (1+2\beta)\int_0^{\zeta_0} F^2\, d\zeta - \int_0^{\zeta_0} G^2\, d\zeta \\ -G'(0) &= (2+2\beta)\int_0^{\zeta_0} FG\, d\zeta \end{aligned}\right\} \tag{I-2}$$

Upon substituting (I-1) into them, the system of equations is obtained to define λ and ζ_0

$$\lambda = -(1+2\beta)[0{\cdot}0301\,\lambda^2 - 0{\cdot}00675\,\lambda + 0{\cdot}000397]\,\zeta_0^4 + 0{\cdot}2357$$
$$\zeta_0^4 = 3 : [2(2+\beta)(0{\cdot}0607\,\lambda - 0{\cdot}00564)] \tag{I-3}$$

whence we have for λ which has the physical meaning,

$$\lambda = [1{\cdot}991 + 0{\cdot}675a + \sqrt{(0{\cdot}737 + 0{\cdot}1234a - 0{\cdot}02236a^2)}] : (12{\cdot}14 + 6{\cdot}02a)$$

where

$$a = \frac{3}{4}\,\frac{1+2\beta}{1+\beta} \tag{I-4}$$

When λ is defined by this equation, it is possible to find ζ_0 and the corresponding values of the unknown quantities

$$F'(0) = \lambda\,\zeta_0 \quad G'(0) = -\tfrac{3}{2}\,\zeta_0^{-1} \tag{I-5}$$

Ratios of these quantities to the appropriate values for $\beta = 1$ (rotating disk)

$$\frac{F'(0;\beta)}{F'(0;1)} = \frac{\lambda\,\zeta_0}{(\lambda\,\zeta_0)_{\beta=1}}, \qquad \frac{G'(0;\beta)}{G'(0;1)} = \frac{(\zeta_0)_{\beta=1}}{\zeta_0} \tag{I-6}$$

agree well with exact ones up to $\beta \approx 2$. Therefore, if $F'(0; 1)$ and $G'(0; 1)$ are taken from [7], rather a good approximation for $F'(0)$ and $G'(0)$ from $\beta = 1$ to $\beta = 2$ is found by these formulae.

APPENDIX II

Some Solutions of Thermal Boundary-layer Equations at $Pr = 1$

At $Pr = 1$ the second equation of system (13) becomes:

$$\beta\sigma' H + 2F\sigma = \sigma'' + (F'^2 + G'^2) \qquad \text{(II-1)}$$

If the first two equations of system (4) multiplied by F and $(G + K)$, respectively, are added to this equation, we have:

$$H\beta(\sigma + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 + KG) + 2F(\sigma + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 + KG) = (\sigma + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2 + KG) \qquad \text{(II-2)}$$

The solution of this equation is

$$\sigma + \tfrac{1}{2}(F^2 + G^2) + KG = 0 \qquad \text{(II-3)}$$

where K is defined from the boundary conditions

For the isothermal surface $\sigma(0) = \sigma(\infty) = 0$, therefore, $K = -\frac{1}{2}$, i.e.

$$\sigma = \tfrac{1}{2}G - \tfrac{1}{2}(F^2 + G^2) \qquad \text{(II-4)}$$

For the surface thermally insulated $\sigma'(0) = \sigma(\infty) = 0$, therefore, $K = -1$, i.e.

$$\sigma_i = G - \tfrac{1}{2}(F^2 + G^2) \qquad \text{(II-5)}$$

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Abstract—Results of calculations are obtained for similar boundary layers on axisymmetric surfaces rotating in an infinite motionless medium. An electronic computer was employed. The power dependence of the distance from the rotation axis on the length of the generating line is the condition of the existence of similar layers.

Characteristics of velocity and thermal boundary layers, velocity and temperature distributions are calculated.·

Using the class of exact solutions obtained, an approximate method for calculating velocity and thermal boundary layers on arbitrary-shaped rotating surfaces is developed. The method employed is to choose on each section a closely approximate surface with power dependence of radius on the component length; taking into account the continuous consolidation of the boundary layer.

Using, as an example the case of a rotating sphere, it is shown that the calculation results obtained with this method agree with both the data of other calculations and experiment.

Résumé—Les résultats des calculs sont obtenus pour des couches limites en similitude sur des surfaces de révolution tournant dans un milieu infini immobile. Un calculateur électronique a été employé. La variation de la distance de l'axe de rotation sous la forme d'une puissance de la longueur de la génératrice est la condition d'existence de couches en similitude.

Les caractéristiques des couches limites dynamiques et thermiques, les distributions de vitesse et de température sont calculées.

En employant la classe de solutions exactes obtenue, une méthode approchée pour calculer les couches limites dynamiques et thermiques sur des surfaces tournantes de forme arbitraire est exposée. La méthode employée consiste à choisir, sur chaque section, une surface approchée avec une variation du rayon selon une puissance de la longueur, en tenant compte de l'augmentation continue de la couche limite.

En prenant comme exemple le cas de la sphère en rotation, on montre que les résultats des calculs obtenus avec cette méthode sont en accord à la fois avec ceux d'autres calculs et ceux de l'expérience.

Zusammenfassung—Für gleichartige Grenzschichten auf achsensymmetrischen Oberflächen, die in einem unendlichen, ruhigen Medium rotieren, werden mit einem Elektronenrechner Ergebnisse erstellt. Die Bedingung für das Vorhandensein gleichartiger Grenzschichten ist die Potenzabhängigkeit des Achsabstandes von der Länge der erzeugenden Strecke.

Berechnet werden Kenngrössen von Geschwindigkeit und thermischer Grenzschicht und die Geschwindigkeits- und Temperaturverteilung. Unter Benützung der erzielten, genauen Lösungen wird eine Näherungsmethode zur Berechnung der Geschwindigkeit und der thermischen Grenzschicht auf beliebig gestalteten rotierenden Oberflächen entwickelt. Die verwendete Methode wählt für jeden Abschnitte eine sehr stark angenäherte Oberfläche, deren Radius von dem Längenanteil über eine Potenz abhängig ist. Dabei wird eine dauernde Ausbildung der Grenzschicht mit berücksichtigt.

Am Beispiel für den Fall der rotierenden Kugel wird gezeigt, dass die mit dieser Methode erzielten Rechenergebnisse sowohl mit den Werten anderer Berechnungen als auch mit experimentellen Messungen übereinstimmen.